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ON RAMIFICATIONS OF ARTIN-SCHREIER EXTENSIONS OF SURFACES OVER ALGEBRAICALLY CLOSED FIELDS OF POSITIVE CHARACTERISTIC II

MASAO OI

ABSTRACT. For a smooth surface X over an algebraically closed field of positive characteristic, we consider the ramification of an Artin-Schreier extension of X . A ramification at a point of codimension 1 of X is understood by the Swan conductor. A ramification at a closed point of X is understood by the invariant r_x defined by Kato [Amer. J. Math. **116**]. The main theme of this paper is to construct the Young diagram $Y(X, D, x)$ which is closely related to r_x and to prove Kato's conjecture [Amer. J. Math. **116**] for an upper bound of r_x for a good Artin-Schreier extension.

1. INTRODUCTION

Let F be an algebraically closed field of characteristic $p > 0$. Let X be a smooth proper surface over F , D a simple normal crossing divisor. Put $U = X - D$. For a closed point $x \in D$, we call Case (I) if the number of irreducible components of D containing x is one, we call Case (II) if the number of irreducible components of D containing x is two. We also say x is of type (I) or type (II). In this paper, we always assume D is generated by t_1 in $O_{X,x}$ in case (I), and D is generated by $t_1 t_2$ in $O_{X,x}$ in case (II), where $t_1, t_2 \in O_{X,x}$. Throughout this paper, we fix x once and for all.

Let l be a prime number which is different from p . For a character $\chi : \pi_1(U) \rightarrow \bar{\mathbb{Q}}_l^\times$ of order p , Kato has defined an invariant $r_x = r_x(\chi)$ in his paper [2]. The invariant r_x is related to the Euler Poincaré characteristic of \mathfrak{F}_χ , where \mathfrak{F}_χ is the étale sheaf corresponding to χ . Let K be the function field of X , and K' be the Artin-Schreier extension of K corresponds to χ .

By the Artin-Schreier theory, there is an element $f \in K$ such that $K' = K(\alpha)$ and $\alpha^p - \alpha = f$. f is determined by K'/K modulo $\beta(K)$, where β is the Artin-Schreier map $x \mapsto x^p - x$.

In part I of this paper ([3]), we studied certain Artin-Schreier extensions of 2-dimensional affine plane over F and found an algorithm to compute r'_x which is equal to r_x for “almost all” extensions. In this paper, we generalize this result to any Artin-Schreier extension of surfaces over F . Moreover, we associate a Young diagram to the Artin-Schreier extension K' , and give an upper bound of r_x .

2. DEFINITION OF r_x , r'_x , CLEAN MODELS, AND SWAN CONDUCTOR

Let the notation be as in the introduction. We recall the definition of a clean model following [3].

Definition 1. (Case I) We say (X, U, χ) is clean at $x \in D$ if there exists $g \in K$ such that $g - f \in \beta(K)$ and one of the following holds.

(I-1) $g \in O_{X,x}$.

(I-2) $g = u/t_1^n$ ($u \in O_{X,x}^\times$, $n \geq 1$, $\gcd(n, p) = 1$).

(I-3) $g = t/t_1^n$ ($n \geq 1$, $t \in O_{X,x}$ and (t_1, t) is the maximal ideal).

(Case II) We say (X, U, χ) is clean at $x \in D$ if there exists $g \in K$ such that $g - f \in \beta(K)$ and one of the following holds

(II-1) $g \in O_{X,x}$.

(II-2) $g = u/t_1^a t_2^b$, ($u \in O_{X,x}^\times$, $a, b \geq 1$ and $\gcd(a, b, p) = 1$).

Recall that the Swan conductor $Sw_D(f)$ is defined by

$$Sw_D(f) := \min\{\max\{-v_D(g), 0\} \mid g \in K, g \equiv f \pmod{\beta(K)}\},$$

where v_D is the normalized additive valuation on K defined by D . Let $X' = X_s \rightarrow X_{s-1} \rightarrow \cdots \rightarrow X_0 = X$ be a sequence of blowing-ups of closed points lying over x such that (X', U', χ) is clean at all points of

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$X' \setminus U'$ with U' the inverse image of U in X' . For each $0 \leq i < s$, let μ_i be the following nonnegative integer. Let U_i be the inverse image of U in X_i . Let $\mu_i = e_i(e_i - 1)$ in Case (I) (resp. $\mu_i = e_i^2$ in Case (II)) with $e_i \geq 0$ the integer defined with respect to the blowing up $pr_i : X_{i+1} \rightarrow X_i$ at x_i . Here

$$e_i := \begin{cases} Sw_{D_{1,i}}(f) - Sw_{pr_i^{-1}(x_i)}(f) & \text{if } x_i \text{ is of type (I),} \\ Sw_{D_{1,i}}(f) + Sw_{D_{2,i}}(f) - Sw_{pr_i^{-1}(x_i)}(f) & \text{if } x_i \text{ is of type (II),} \end{cases}$$

where $D_{1,i}$ (resp. $D_{1,i}, D_{2,i}$) is the irreducible divisors of X_i containing x_i . The invariant r_x is defined by

$$r_x = \sum_{i=0}^s \mu_i.$$

In Case (I), we choose t_2 such that (t_1, t_2) is the maximal ideal of $O_{X,x}$. We fix t_2 once and for all if x is of type (I).

We may assume $f \in O_{X,x}[(t_1 t_2)^{-1}]$. This can be seen as follows. Let V be an affine neighbourhood of x and put $V' = V - D$. Then we have an Artin-Schreier exact sequence

$$0 \rightarrow \mathbb{Z}/p\mathbb{Z} \rightarrow \Gamma(V', O_{V'}) \rightarrow \Gamma(V', O_{V'}) \rightarrow H_{et}^1(V', \mathbb{Z}/p\mathbb{Z}) \rightarrow H^1(V', O_{V'}) = 0.$$

The character χ can be considered as an element of $H_{et}^1(V', \mathbb{Z}/p\mathbb{Z})$. Therefore χ is the image of an element $f \in \Gamma(V', O_{V'}) \subset O_{X,x}[(t_1 t_2)^{-1}]$. In fact, the proof above shows that we may assume $f \in O_{X,x}[t_1^{-1}]$ in Case (I).

Lemma 1. *Let A be a regular local ring of dimension 2 with maximal ideal (y_1, y_2) . Let g be an element of $A[(y_1 y_2)^{-1}]$. Then, there exist integers m, n , $0 < a_1 < \dots < a_k$, $0 < b_1 < \dots < b_k$ and $c_0, \dots, c_k \in A^\times$ such that*

$$g = y_1^{-n} y_2^m c_0 + y_1^{-n-a_1} y_2^{m+b_1} c_1 + \dots + y_1^{-n-a_k} y_2^{m+b_k} c_k.$$

Moreover, the integer $m, n, a_1, \dots, a_k, b_1, \dots, b_k$ are uniquely determined.

PROOF. It is well-known that every regular local ring is a UFD. By multiplying some power of $y_1 y_2$, we may assume that $g \in A$. We can choose an integer $i \geq 0$ such that $g = y_1^i g_1$, $g_1 \notin (y_1)$. Since the ring $\bar{A} = A/(y_1)$ is a DVR, there exists an integer $j \geq 0$ such that $\bar{g}_1 = \bar{y}_2^j \bar{g}_2'$, with $\bar{g}_2' \in \bar{A}^\times$. Here, \bar{g}_1 and \bar{y}_2 are images of g_1 and y_2 in \bar{A} , respectively. Moreover $A^\times \rightarrow \bar{A}^\times$ is surjective, since $A^\times = A - (y_1, y_2)A$ and $\bar{A}^\times = \bar{A} - \bar{y}_2 \bar{A}$. Therefore there exist an integer $j_i \geq 0$ and $g_2 \in A^\times$ such that $g - y_1^i y_2^{j_i} g_2 \in (y_1)^{i+1}$. Repeating this argument, we can find $c'_i \in A^\times$, $j_i \in \mathbb{Z}_{\geq 0}$ for each $i \geq 0$ such that

$$g = \sum_{i=0}^{\infty} c'_i y_1^i y_2^{j_i}$$

in the completion \hat{A} of A . Let $j_\infty = \min\{j_i | i \in \mathbb{Z}_{\geq 0}\}$, and s the smallest integer such that $j_s = j_\infty$. Put

$$\begin{aligned} g(s) &:= g - \sum_{i=0}^{s-1} c'_i y_1^i y_2^{j_i} \\ &= \sum_{i=s}^{\infty} c'_i y_1^i y_2^{j_i} \in y_1^s y_2^{j_s} \hat{A}. \end{aligned}$$

Since $y_1^s y_2^{j_s} \hat{A} \cap A = y_1^s y_2^{j_s} A$, we have $g(s) \in y_1^s y_2^{j_s} A$. Put

$$L = \{(i, j(i)) \in \mathbb{Z}_{\geq 0}^2 \mid c(i) \neq 0\} + \mathbb{Z}_{\geq 0}^2.$$

Let $\{(\alpha_0, \beta_0), (\alpha_1, \beta_1), \dots, (\alpha_k, \beta_k)\}$ be the set of minimal generators of L by the action of monoid $\mathbb{Z}_{\geq 0}^2$ such that $\alpha_0 < \alpha_1 < \dots < \alpha_k$. In other words, $\{(\alpha_i, \beta_i) \mid i = 0, \dots, k\}$ is the set of the minimal elements of L , where minimality is considered with respect to the partial order defined by $(a, b) < (a', b') \Leftrightarrow a < a', b < b'$.

Thus we have

$$\begin{aligned} j_{\alpha_0} &= \beta_0, & j_{\alpha_0+1}, \dots, j_{\alpha_1-1} &\geq \beta_0 > j_{\alpha_1} = \beta_1, \\ j_{\alpha_i} &= \beta_i, & j_{i+1}, \dots, j_{\alpha_{i+1}-1} &\geq \beta_i > j_{\alpha_{i+1}} = \beta_{i+1}, & (i = 1, \dots, k-1) \\ j_s &= \beta_k. \end{aligned}$$

Put $c_i = \sum_{l=\alpha_i}^{\alpha_i-1} c'_l y_1^{l-\alpha_i} y_2^{j_l-\beta_i}$ ($i = 0, \dots, k-1$) and $c_k = y_1^{-s} y_2^{-j_s} g(s)$. Then we have $c_i \in A^\times$ and

$$g = \sum_{i=0}^k c_i y_1^{\alpha_i} y_2^{\beta_i}$$

as required.

For any $\mathbb{Z}_{\geq 0}^2$ -stable subset $R \subset \mathbb{Z}_{\geq 0}^2$, we denote \mathfrak{M}_R the ideal of A generated by $\{y_1^i y_2^j \mid (i, j) \in R\}$. Then clearly L is the minimal $\mathbb{Z}_{\geq 0}^2$ -stable subset of $\mathbb{Z}_{\geq 0}^2$ such that $f \in \mathfrak{M}_L$. Thus the set $\{(\alpha_0, \beta_0), \dots, (\alpha_k, \beta_k)\}$ is determined by f . \square

By Lemma 1, there is an expression

$$f = t_1^{-n} t_2^m c_0 + t_1^{-n-a_1} t_2^{m+b_1} c_1 + \dots + t_1^{-n-a_k} t_2^{m+b_k} c_k, \quad c_i \in O_{X,x}^\times (i = 1, \dots, k).$$

We put $a_0 = b_0 = 0$. We define $\text{pg}(f)$, $\text{pg}_1(f)$, $\overline{\text{pg}}(f)$, $\overline{\text{pg}}_1(f)$, and $\overline{\text{pg}}_{\text{inv}}(f)$ as follows.

$$\begin{aligned} \text{pg}(f) &:= ((n, m), (n + a_1, m + b_1), \dots, (n + a_k, m + b_k)) \in \oplus_{0 \leq i \leq k} \mathbb{Z}^2, \\ \overline{\text{pg}}(f) &:= \{(n + a_i, m + b_i) \mid 0 \leq i \leq k\} \subset \mathbb{Z}^2, \\ \overline{\text{pg}}_{\text{inv}}(f) &:= \{(-n - a_i, m + b_i) \mid 0 \leq i \leq k\} \subset \mathbb{Z}^2. \end{aligned}$$

When $A = O_{X,x}$, we often write $\text{pg}_x(f)$ instead of $\text{pg}(f)$. Note that the completion $\hat{O}_{X,x}$ can be naturally identified with $F[[t_1, t_2]]$. The element $f \in O_{X,x}[[t_1 t_2]^{-1}]$ has a formal power series expansion

$$f = \sum_{i,j \in \mathbb{Z}} \gamma_{i,j} t_1^i t_2^j, \quad \gamma_{i,j} \in F.$$

We put $L_f = \{(i, j) \in \mathbb{Z}^2 \mid \gamma_{i,j} \neq 0\} + \mathbb{Z}_{\geq 0}^2$. Let $\{(\alpha_i, \beta_i) \mid i = 0, \dots, k\}$ be the set of minimal elements of L_f . By Lemma 1, we have $\overline{\text{pg}}_{\text{inv}}(f) = \{(\alpha_i, \beta_i) \mid i = 0, \dots, k\}$.

Lemma 2. *Let B be a DVR of characteristic p over a perfect field F_1 . Assume that the residue field of B is canonically isomorphic to F_1 , and that the fractional field K_B of B is of transcendental degree 1 over F_1 . Let \hat{B} be a completion of B with respect to the maximal ideal. Then we have $B \cap \hat{B}^p = B^p$.*

PROOF. Let t be a prime element of B . It is well known that B is a free B^p -module of rank p with basis $\{1, t_1, \dots, t_1^{p-1}\}$ since $[K_B : K_B^p] = 1$. It is also well-known that \hat{B} is a free \hat{B}^p -module of rank p with the same basis. Hence the lemma follows. \square

Lemma 3. *Let $g \in O_{X,x}[(t_1 t_2)^{-1}]$ be an element with $\text{Sw}_{(t_1)}(g) = n > 0$. Then there exists an element $g' \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ with the following properties:*

- (a) $\min\{i \mid (i, j) \in L_{g'}\} = -n$.
- (b) $p \nmid \gcd(n, b)$, where b is the smallest integer such that $(-n, b) \in L_{g'}$.

Here, $\beta : K \rightarrow K$ is the Artin-Schreier map.

PROOF. By the definition of the Swan conductor, there exists an element $g' \in K$ such that $v_{(t_1)}(g) = -n$. One can easily show that g' can be chosen in $O_{X,x}[(t_1 t_2)^{-1}]$. Then we have $\min\{i \mid (i, j) \in L_{g'}\} = -n$. If $p \nmid n$, the condition (b) is empty, and so we may assume that $p \mid n$. We shall show that there exists b such that $p \nmid b$ and $(-n, b) \in L_{g'}$. By Lemma 2 applied for $B = O_{X,x}/(t_1)$, there exists $g_1 \in O_{X,x}[t_2^{-1}]$ such that $t_1^n g' \equiv g_1^p \pmod{t_1 O_{X,x}[t_2^{-1}]}$. Put $g_2 = g' - t_1^{-n} g_1^p + t_1^{-n/p} g_1$. Then $g_2 \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ and $\text{ord}_{(t_1)}(g_2) > -n$, a contradiction. It follows that there exists $b' \in \mathbb{Z}$ such that $p \nmid b'$ and $(-n, b') \in L_{g'}$. Now we consider the expansion of g' .

$$g' = \sum_{i,j \in \mathbb{Z}} \epsilon_{i,j} t_1^i t_2^j, \quad \epsilon_{i,j} \in F.$$

Put

$$g'' = g' - \sum_{\substack{(-n,j) \in L_{g'} \\ j < b'}} \left(\epsilon_{i,j} t_1^{-n} t_2^j - \epsilon_{i,j}^{1/p} t_1^{-n/p} t_2^{j/p} \right).$$

Then g'' has the desired properties. \square

Lemma 4. *Let $g \in O_{X,x}[(t_1 t_2)^{-1}]$ be an element with $Sw_{(t_1)}(g) = n > 0$. Suppose that $g' \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ and $-m := \min\{i \mid (i, j) \in L_{g'}\} < -n$. Then p divides m and all integer b such that $(-m, b) \in L_{g'}$.*

PROOF. As we have seen in the proof of Lemma 3, there exists $g_1 \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ such that $\min\{i \mid (i, j) \in L_{g_1}\} = -n$. Then we have $g' - g_1 \in \beta(O_{X,x}[(t_1 t_2)^{-1}])$. The lemma follows from this. \square

Lemma 5. *Let $g \in O_{X,x}[(t_1 t_2)^{-1}]$ be an element with $Sw_{(t_1)}(g) = n_1 > 0$ and $Sw_{(t_2)}(g) = n_2 > 0$. Then there exists an element $g' \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ with the following properties:*

- (a) $\min\{i \mid (i, j) \in L_{g'}\} = -n_1$.
- (b) $p \nmid \gcd(n_1, b_1)$, where b_1 is the smallest integer such that $(-n_1, b_1) \in L_{g'}$.
- (c) $\min\{j \mid (i, j) \in L_{g'}\} = -n_2$.
- (d) $p \nmid \gcd(n_2, b_2)$, where b_2 is the smallest integer such that $(-n_2, b_2) \in L_{g'}$.

PROOF. By Lemma 3, there exists $g' \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ with properties (c) and (d). Suppose that $-m := \min\{i \mid (i, j) \in L_{g'}\} < -n$. By Lemma 4, p divides m and all integer b such that $(-m, b) \in L_{g'}$. As in the proof of Lemma 3, one can find an element $g_1 \in O_{X,x}[t_2^{-1}]$ such that $t_1^{n_1} g' \equiv g_1^p \pmod{t_1 O_{X,x}[t_2^{-1}]}$. We may assume $v_{(t_2)}(g_1) \geq -n_2/p$. Put $g_2 = g' - t_1^{-n} g_1^p + t_1^{-n/p} g_1$. Then we have $v_{(t_2)}(g_2) \geq v_{(t_2)}(g')$. Moreover, g_2 has also properties (a) and (b). Repeating this argument, one can find an element $g' \in g + \beta(O_{X,x}[(t_1 t_2)^{-1}])$ with properties (a), (b), (c), and (d). The rest of the proof is the same as that of Lemma 3. \square

Lemma 6. *Suppose that $g \in O_{X,x}[(t_1 t_2)^{-1}]$ has an expression*

$$g = t_1^{-n} t_2^m c_0 + t_1^{-n-a_1} t_2^{m+b_1} c_1 + \cdots + t_1^{-n-a_k} t_2^{m+b_k} c_k, \quad c_i \in O_{X,x}^\times (i = 1, \dots, k).$$

We put $L_g := \overline{\text{pg}}_{\text{inv}}(g) + \mathbb{Z}_{\geq 0}^2$. Then there exists $g' \in g + \beta(K)$ such that $(0, 0) \in L_{g'}$, and $\gcd(a, b, p) = 1$ for all non-zero $(a, b) \in \overline{\text{pg}}_{\text{inv}}(g')$.

PROOF. We first consider Case (I). By Lemma 3, we may assume that $-n - a_k = Sw_{(t_1)}(g)$ and that $p \nmid (Sw_{(t_1)}(g), m + a_k)$. Put $n' = \min\{n, 0\}$. Using the identity $c_{i,j} t_1^{-pa'} t_2^{-pb'} - c_{i,j}^{1/p} t_1^{-a'} t_2^{-b'} \in \beta(K)$ for any $c_{i,j} \in F$, we see that there exists $g' \in g + \beta(K)$ such that $\overline{\text{pg}}_{\text{inv}}(g') \cap p\mathbb{Z}^2 \subset \{(0, 0)\}$ and $(0, 0) \in L_{g'}$. Note that we only use the above identity for $(-pa', -pb') \in [-Sw_{(t_1)}(g), -n'] \times [0, m + a_k]$. This settles the proof for Case (I).

Next, we consider Case (II). In this case, there exists $g' \in g + \beta(K)$ such that

$$\overline{\text{pg}}(g') = ((n(g'), -Sw_{t_2}(g)), \dots, (Sw_{(t_1)}(g), m(g') + a_k(g'))).$$

By Lemma 3, we see $(n(g'), -Sw_{t_2}(g))$ and $(Sw_{(t_1)}(g), m(g') + a_k(g'))$ are not divisible by p for the above g' . Using the identity $c_{i,j} t_1^{-pa'} t_2^{-pb'} - c_{i,j}^{1/p} t_1^{-a'} t_2^{-b'} \in \beta(K)$ for any $c_{i,j} \in F$, we see that there exists $g' \in g + \beta(K)$ such that $\overline{\text{pg}}_{\text{inv}}(g') \cap p\mathbb{Z}^2 \subset \{(0, 0)\}$ and $(0, 0) \in L_{g'}$. Note that we only use the above identity for $(-pa', -pb') \in [-Sw_{(t_1)}(g), -n(g')] \times [-Sw_{(t_2)}(g), m(g') + a_k(g')]$. \square

By Lemma 6, we may assume $(n + a_i, m + b_i, p) = 1$ for all non-zero elements $(n + a_i, m + b_i) \in \text{pg}(g)$. We put $(a'_i, b'_i) := (-n - a_i, m + b_i)$ for $i = 0, \dots, k$.

Definition 2. A good representative of the Artin-Schreier extension K'/K is an element $g' \in K$ such that $K' = K'(\alpha)$, $\alpha^p - \alpha = g'$ and $(0, 0) \in L_{g'}$, and $\gcd(a, b, p) = 1$ for all non-zero $(a, b) \in \overline{\text{pg}}_{\text{inv}}(g)$.

Note that a good representative exists by Lemma 6. We will define r'_t ($t = 1$ or $t = 2$ according as x is of type (I) or type (II)) by

$$r'_t((a'_i, b'_i)_{0 \leq i \leq k}) := \mu + r'_2((a'_j, b'_j - a'_j)_{j \in J'_a}) + r'_t((a'_j - b'_j, b'_j)_{j \in J'_b}),$$

where

$$J'_a := \{j \mid b'_j < \max_{j+1 \leq i \leq k} \{b'_i\}\},$$

$$J'_b := \{j \mid a'_j > \max_{0 \leq i \leq j-1} \{a'_i\}\}.$$

Recall that $\mu = e(e-1)$ in Case (I) (resp. $\mu = e^2$ in Case (II)) with $e := \max\{n + a_k, 0\} + \max\{-m, 0\} - \max\{n - m + a_i - b_i\}_{0 \leq i \leq k}, 0\} \geq 0$. This is the analogy of Formula 5 of our previous paper [3]. Furthermore, the following lemma holds.

Proposition 1. Let $\text{pg}(f) = ((n, m), (n + a_1, m + b_1), \dots, (n + a_k, m + b_k))$. We put

$$\text{pg}_1(f) := ((a_1, b_1), \dots, (a_k, b_k)) \in \oplus_{1 \leq i \leq k} \mathbb{Z}^2.$$

If $n + a_i \leq 0$ and $m + b_i \leq 0$ for some i ($0 \leq i \leq k$), then $r'_t(\text{pg}(f))$ depends only on $\text{pg}_1(f)$.

PROOF. We consider the case (I). Recall that we have defined $e_0 = \max\{n + a_k, 0\} + \max\{-m, 0\} - \max\{n - m + a_i - b_i\}_{0 \leq i \leq k}$. Since $n + a_i \leq 0$, and $m + b_i \leq 0$, we see $e_0 = a_k - \max\{a_i - b_i\}_{0 \leq i \leq k}$. It follows that r'_t depends only on $\text{pg}_1(f)$. One can treat Case (II) similarly. \square

Note that the assumption of the Proposition 1 is essential, since $\overline{\text{pg}}(g) \cap \{(x, y) \in \mathbb{Z}^2 \mid x \leq 0, y \geq 0\} \neq \emptyset$ when g is good representative.

Proposition 2. If $n + a_{i'} \leq 0$, $m + b_{i'} \geq 0$ for some i' ($0 \leq i' \leq k$), then $r'_t((a_i, b_i)_{0 \leq i \leq k}) = r'_t((a_i, b_i)_{0 \leq i \neq i' \leq k})$,

PROOF. This easily follows from the fact that the region $\{(x, y) \in \mathbb{Z}^2 \mid x \leq 0, y \geq 0\}$ is preserved by the transformation from (a, b) to $(a, b - a)$ and (a, b) to $(a - b, b)$. \square

Lemma 7. The equality $r'_t(\text{pg}(g)) = r'_t(\text{pg}(g'))$ holds, if $g - g' \in \beta(K)$, $\text{pg}(g) \cap p\mathbb{Z}^2$, and $\text{pg}(g') \cap p\mathbb{Z}^2 \subset \{(0, 0)\}$.

PROOF. We treat the case (II), the proof of case (I) is similar. Note that

$$\begin{aligned} e &:= \max\{n + a_k, 0\} + \max\{-m, 0\} - \max\{n - m + a_i - b_i\}_{0 \leq i \leq k}, 0\} \\ &= \max\{n' + a'_{k'}, 0\} + \max\{-m', 0\} - \max\{n' - m' + a'_{i'} - b'_{i'}\}_{0 \leq i' \leq k'}, 0\}, \end{aligned}$$

for good representatives g and g' . Since r'_2 is the sum of e by the blow-up of the point of type (II), we obtain the equality of this lemma. \square

Definition 3. For an Artin-Schreier extension χ , we put $r'_x(\chi) = r'(\text{pg}(g))$ for a good representative g .

This definition does not depend on the choice of a good representative g by Lemma 7.

3. CONSTRUCTION OF THE YOUNG DIAGRAM

The purpose of this section is to construct the morphism of sets $Y(X, D, x)$. Let X , D , and x be as in the introduction. we construct the map.

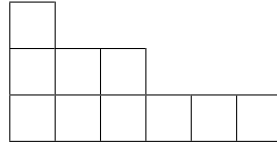
$$Y(X, D, x) : \{ \text{Artin-Schreier extension } \chi \text{ ramified only on } D \} \rightarrow \{ \text{Young diagram} \}$$

This is the composite of pg and Y_t .

We denote the rectangle $[a, b] \times [c, d] \subset \mathbb{R}^2$ by $R(a, b; c, d)$ for $a < b$ and $c < d$. Recall that a Young diagram corresponding to a partition $(\xi_1, \xi_2, \dots, \xi_k)$ with $\xi_1 \geq \xi_2 \geq \dots \geq \xi_k \in \mathbb{Z}_{\geq 0}$. The area $|Y|$ is defined by $\xi_1 + \dots + \xi_k$. We identify the Young diagram corresponding to the partition $(\xi_1, \xi_2, \dots, \xi_k)$ with the set

$$\bigcup_{i=1}^k R(i-1, i; 0, \xi_i) \subset \mathbb{R}^2.$$

Note that larger rows will be below in our custom. For example, the Young diagram corresponding to the partition $(6, 3, 1)$ is



We construct the Young diagram Y associated to an Artin-Schreier extension. Let f and $\text{pg}(f) = ((n, m), (n + a_1, m + b_1), \dots, (n + a_k, m + b_k))$ be as above. In this paper, we always assume $m + a_i \leq 0$, $n + b_i \leq 0$ for some i . We denote the rectangle $[a, b] \times [c, d]$ by $R(a, b; c, d)$ for $a < b, c < d$. We can construct the figure (Young diagram) in the rectangle with vertex $R(0, a_k; 0, b_k - 1)$ in Case (I) (resp. $R(0, a_k; 0, b_k)$ in Case (II)). This is done by the induction.

$\mu = e(e - 1)$ (resp. $\mu = e^2$) is the area of $R(0, e; 0, e - 1)$ in Case (I) ($R(0, e; 0, e)$ in Case (II)). Recall $e := \max\{n + a_i\} - m - \max\{n - m + a_i - b_i\} = a_k - \max\{a_i - b_i\}$. As usual, for the set $S \subset \mathbb{R}^2$ we denote $(a, b) + S := \{(a, b) + s \mid s \in S\}$. We inductively define Y_t as follows.

In the case $k = 0$, we put $Y_t(n, m) := \{(0, 0)\}$. In the case $k > 0$, we define $Y_t((a'_i, b'_i)_{0 \leq i \leq k})$ by the induction on k and depth. Here depth of $(a_i, b_i)_{0 \leq i \leq k}$ is defined by $a'_k - a'_0 + b'_k - b'_0$. This definition is slightly different from our previous paper.

$$\begin{aligned} Y_t((a'_i, b'_i)_{0 \leq i \leq k}) &:= R(0, e; 0, e - (2 - t)) \\ &\cup (0, e - (2 - t)) + Y_2((a'_{j_i}, b'_{j_i} - a'_{j_i})_{i \in J'_a}) \\ &\cup (e, 0) + Y_t((a'_{j_i} - b'_{j_i}, b'_{j_i})_{i \in J'_b}). \end{aligned}$$

Lemma 8. $Y_t((a'_i, b'_i)_{0 \leq i \leq k})$ is a Young diagram in $R(0, a_k; 0, b_k - 1)$ in Case (I) (resp. $R(0, a_k; 0, b_k)$ in Case (II)). where $t = 1$ or $t = 2$ according as x is of type (I) or of type (II).

PROOF. This lemma is proved by induction. By the induction assumption $Y_2((a'_{j_i}, b'_{j_i} - a'_{j_i})_{i \in J'_a}) \subset R(0, e; 0, b_k - e)$. $Y_t((a'_{j_i} - b'_{j_i}, b'_{j_i})_{i \in J'_b}) \subset R(0, e; a_k - e, e)$. \square

We define Y as follows.

$$Y(X, D, x)(\chi) = Y_t \circ \text{pg}(g),$$

where g is a good representative of χ .

Proposition 3. The set $Y(X, D, x)(\chi)$ does not depends on the choice of good representative g .

PROOF. The proof of this lemma is completely the same as that of Lemma 7. \square

4. PROPERTIES OF $Y(X, D, x)$

The map $Y(X, D, x)$ has the following properties. The following theorem is our first main theorem.

Theorem 4.1. $r'_x(\chi)$ is equal to $|Y(X, D, x)(\chi)|$.

PROOF. This is proved by the induction of k and depth. By the induction assumption

$$\begin{aligned} |Y_t((a'_i, b'_i)_{0 \leq i \leq k})| &:= \mu + |Y_2((a'_{j_i}, b'_{j_i} - a'_{j_i})_{i \in J'_a})| + |Y_t((a'_{j_i} - b'_{j_i}, b'_{j_i})_{i \in J'_b})| \\ &= \mu + r'_2((a'_j, b'_j - a'_j)_{j \in J'_a}) + r'_t((a'_j - b'_j, b'_j)_{j \in J'_b}) \\ &= r'_t((a'_i, b'_i)_{0 \leq i \leq k}) \end{aligned}$$

as required. \square

As a consequence of the above theorem, we prove Kato's conjecture [2] for upper bound of r_x under some assumption. To state this more preciously, we introduce the notion good Artin-Schreier extension.

Definition 4. Let f be a good representative for an Artin-Schreier extension K'/K . We say that a line $L \subset \mathbb{R}^2$ is a special line for $\overline{\text{pg}}(f)$ if L satisfies either of the following conditions (G1) or (G2):

- (G1) The line L contains three points of $\overline{\text{pg}}(f)$. Moreover, all points of $\overline{\text{pg}}(f)$ belong to one side of L .
- (G2) The line L contains two points $(a'_i, b'_i), (a'_j, b'_j) \in \overline{\text{pg}}(f)$, $(0 \leq i < j \leq k)$ such that $p \mid \gcd(a'_j - a'_i, b'_j - b'_i)$. Moreover, all points of $\overline{\text{pg}}(f)$ belong to one side of L .

We denote by $I_L(f)$ the set of all special lines for $\overline{\text{pg}}(f)$. We say that an Artin-Schreier extension K'/K is good if there are no special lines for $\overline{\text{pg}}(f)$ for some good representative f .

Example 1. Let $K_1 = K(x, y)$, and let m, n be positive integers. The simplest examples of good Artin-Schreier extensions are $K_1(\alpha_{m,n})/K_1$, where $\alpha_{m,n}^p - \alpha_{m,n} = y^m/x^n$.

The following theorem is our second main theorem.

Theorem 4.2. $r_x(\chi) = r'_x(\chi)$ for a good Artin-Schreier extension χ .

PROOF. For simplicity, we consider only the Case (II). By the assumption, we obtain the clean model by successive blow up by points of type (II). This follows from Theorem 4.3 of [3]. In fact, one need a blow up $X'' \rightarrow X'$ by a point $x' \in X'$ of type (I) only when either of the following holds.

- (1-1) $\text{pg}_{x'}(f)$ contains three points of the form (a, b) , $(a + v, b + v)$ and $(a + w, b + w)$ such that $a, b, v, w \in \mathbb{Z}$.
- (2-1) $\text{pg}_{x'}(f)$ contains two points of the form (a, b) and $(a + v, b + v)$ such that $a, b, v \in \mathbb{Z}$ and $p \mid v$.

Note that if the set $(a_i, b_i)_{0 \leq i \leq k}$ has a special line, then so does the set $(a_i, b_i - a_i)_{0 \leq i \leq k}$. Similarly, if the set $(a_i, b_i)_{0 \leq i \leq k}$ has a special line, then so does the set $(a_i - b_i, b_i)_{0 \leq i \leq k}$. The theorem follows from this. \square

5. A PROOF OF KATO'S CONJECTURE FOR GOOD ARTIN-SCHREIER EXTENSIONS

We recall Kato's conjecture ([2]).

Conjecture 1. [Kato] The following inequality holds

$$\begin{aligned} r_x &\leq (n + a_k) \cdot (m + b_k - 1) & x \text{ is of type (I),} \\ r_x &\leq (n + a_k) \cdot (m + b_k) + (-m) \cdot n & x \text{ is of type (II).} \end{aligned}$$

Theorem 5.1. *Conjecture 1 is true if the Artin-Schreier extension K'/K is good.*

PROOF. Using Theorem 4.1 and Theorem 4.2, we see $r_x \leq a_k(b_k - (t - 2))$. In this case Kato's conjecture is refined by our result. \square

6. AN UPPER BOUND OF r_x

Let g be a good representative for χ . We define the irregular term $\text{IrrTerm}_L(g)$ as follows. Suppose that the points (a', b') , (a'', b'') in $\text{pg}(g)$ are contained in the line $L \in I_L(g)$ and that $\text{pg}(g) \cap L \subset [(a', b'), (a'', b'')]$. We put $v_x = a'' - a'$, $v_y = b'' - b'$. Then we set

$$\text{IrrTerm}_L(g) := \max\{(-v_y a' + v_x b') \cdot (d - 1)/d, 0\}.$$

Here we put $d = \gcd(v_x, v_y)$.

If we assume that Conjecture 1 is true when x is of type (I), then the following inequality (1) holds.

$$(1) \quad r_x \leq |Y(X, D, x)(\chi)| + \sum_{L \in I_L(g)} \text{IrrTerm}_L(g).$$

7. AN APPLICATION TO THE EULER-POINCARÉ CHARACTERISTIC

Let \mathfrak{F}_χ be the étale sheaf on U corresponding to χ . We denote by K_X^{\log} the log canonical divisor. From Kato's theory in [4], we have

$$\chi_c(U, \mathfrak{F}_\chi) - \chi_c(U) = (Sw(\chi), Sw(\chi) + K_X^{\log}) - \sum_{x \in X} r_x.$$

Here, χ_c is the compact support étale cohomological Euler-Poincaré characteristic of \mathfrak{F}_χ . From this, we obtain a lower bound of the Euler-Poincaré characteristic for a good Artin-Schreier extension using our inequality $r_x \leq a_k(b_k - (t - 2))$.

For example, let X be 2-dimensional projective plane over F , let D_1, D_2 be two distinct (projective) lines in X , and let $x = D_1 \cap D_2$. We put $U = X \setminus (D_1 \cup D_2)$. Then

$$\chi_c(U, \mathfrak{F}_\chi) \geq \chi_c(U) + (Sw(\chi), Sw(\chi) + K_X^{\log}) - a_k b_k.$$

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